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2006 J. Phys. A: Math. Gen. 39 13861

(http://iopscience.iop.org/0305-4470/39/45/002)

Institute of Physics Publishing Journal of Physics A: Mathematical and General
J. Phys. A: Math. Gen. **39** (2006) 13861–13868 doi:10.1088/0305-4470/39/45/002

Stochastic resonance in a mono-stable system with multiplicative and additive noise

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Received 27 February 2006, in final form 10 September 2006
Published 24 October 2006
Online at stacks.iop.org/JPhysA/39/13861

Abstract
The stochastic resonance in a biased mono-stable system subject to multiplicative and additive noise is investigated. Based on the adiabatic approximation theory, the analytic expression of the signal-to-noise ratio (SNR) is obtained. It is shown that the SNR is a non-monotonic function of the intensities of the multiplicative and additive noise, as well as the parameters of the mono-stable system.

PACS number: 05.40.−a

1. Introduction

The stochastic resonance (SR) phenomenon has been a subject of considerable study in the last two decades [1–29]. SR began with the study of climatic dynamics. In 1981, Benzi and coworkers [1] invoked this phenomenon to explain the Earth's climatic change: the eccentricity of the Earth's orbit varies periodically in time, and if the amplitude of this variation is too small to explain the succession of ice ages and relatively warm periods, the periodical phenomenon is amplified by some perturbations. This cooperative effect between the coherent 'signal' and the 'noise' was called stochastic resonance (SR).

SR has been investigated experimentally and theoretically. McNamara *et al* [2, 3] observed the SR by a bidirectional ring laser and obtained the expression of the signal-to-noise (SNR) in the adiabatic limit. Dykman *et al* [4] and Hu *et al* [5] introduced the linear-response theory and perturbation theory to investigate the SR. Zhou and Moss [6] employed the residence-time distribution to explain the SR as a resonance synchronization phenomenon. The SR is considered as a cooperative result of periodic signal and noise in a nonlinear system. Later, Berdichevsky and Gitterman [7, 8] explored the existence of the SR in a linear system driven by a coloured multiplicative noise or a dichotomous noise. These results have led to an extensive application of the SR in many scientific fields, such as electronic systems [9–11], lasers systems [2], threshold systems [12, 13] and biological systems [14, 15], etc. Several quantifiers have been used to characterize SR in noisy, continuous systems. The average output amplitude, or the spectral amplification (SPA), has been studied in [16, 17] and the phase of the output average in [18–20], respectively. Those parameters as well as the signal-to-noise ratio (SNR) [3] exhibit a non-monotonic behaviour with the noise strength which is a representative of SR. The conventional SR is a nonlinear effect that accounts for the optimum response of a dynamical system to an external force at certain noise intensity. The SR in a broad sense means the non-monotonic behaviour of the output signal as a function of some characteristics of the noise (noise intensity or noise correlation time) or of a periodic force (amplitude or frequency).

In actual systems there are a lot of mono-stable systems [21–29], including chemical, electronic, physical and biological systems. Dykman *et al* [21] and Evstigneev *et al* investigated the SR in a mono-stable over-damped system [23] based on linear response theory. Stocks *et al* investigated the zero-dispersion stochastic resonance (ZDSR) in a mono-stable system [28, 29], for which the dependence of eigenfrequency upon energy has an extremum. They analysed the SR phenomenon on the basis of linear response theory and the fluctuation dissipation theorem and found that the response to a weak periodic force on frequency is strongly resonant. It is well known that the multiplicative noise often plays a different role on the output of a system, with respect to the additive noise. Therefore, the investigation of the response of a mono-stable system driven by the multiplicative noise is of great significance. In this paper, based on the adiabatic approximation theory, we study the SR in a mono-stable system driven by the multiplicative and additive white noise and a periodic force with a constant component. The increase in the constant component leads to a static asymmetry of the mono-stable potential.

2. The mono-stable system and its signal-to-noise ratio

Consider an over-damped mono-stable system [22] with multiplicative and additive noise described by the following Langevin equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -ax^3 + b + x\xi(t) + \eta(t) + A\cos(\Omega t), \tag{1}$$

where $a > 0$, b is a constant, denoting the bias of the mono-stable system. The noise terms $\xi(t)$ and $\eta(t)$ are the uncorrelated noise characterized by their mean and variance

$$\langle \xi(t)\rangle = \langle \eta(t)\rangle = 0, \tag{2}$$

$$\langle \xi(t)\xi(s)\rangle = 2D\delta(t-s), \qquad \langle \eta(t)\eta(s)\rangle = 2P\delta(t-s). \tag{3}$$

Here D and P are the intensities of the multiplicative and additive noise, respectively.

According to equations (1)–(3), the corresponding Fokker–Plank equation of the mono-stable system, equation (1), can be written as

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}[F(x,t)\rho(x,t)] + \frac{\partial^2}{\partial x^2}[G(x)\rho(x,t)], \tag{4}$$

where

$$F(x,t) = Dx - ax^3 + b + A\cos(\Omega t), \qquad G(x) = Dx^2 + P. \tag{5}$$

We assume that the external force frequency Ω is so small that there is enough time for the system to reach the local equilibrium during the period of $1/\Omega$, i.e., we make the assumption that the system satisfies the adiabatic approximation condition [3]. The quasi-stationary distribution function can be derived from equations (4), (5) in the adiabatic limit, i.e.,

$$\rho_{\mathrm{st}}(x) = \frac{C_{\mathrm{st}}}{\sqrt{G(x)}} \exp\left[-\frac{V(x)}{D}\right], \tag{6}$$

where C_{st} is the normalization constant, $V(x)$ is the rectified potential function and has the form

$$V(x) = \int_{-\infty}^{x} \frac{D}{G(x)}[-U'(x) + b + A\cos(\Omega t)]\,\mathrm{d}x, \tag{7}$$

with

$$U'(x) = \frac{\mathrm{d}U}{\mathrm{d}x} = ax^3 - Dx. \tag{8}$$

From equations (7), (8), one can see that, for the case of $D \neq 0$, i.e., in the presence of the multiplicative noise, the mono-stable system (1) can thus be regarded as an equivalent bistable system, with $x_u = 0$ and $x_\pm = \pm\sqrt{D/a}$ being the unstable and stable states of the equivalent bistable system. Under the adiabatic limit condition, the transition rates out of $x_\pm$ can be obtained by

$$\begin{aligned} N_\pm(t) &= \frac{\sqrt{|U''(x_u)U''(x_\pm)|}}{2\pi} \exp\left[\frac{V(x_\pm) - V(x_u)}{D}\right] \\ &= N_{\pm 0} \exp[\mp kA\cos(\Omega t)], \end{aligned} \tag{9}$$

where $N_{\pm 0}$ denotes the characteristic switching frequency of the equivalent bistable system when it is only driven by the multiplicative and additive noise, which is given by

$$N_{\pm 0} = \frac{D}{\sqrt{2}\pi} \exp\left[\mp kb - \frac{\Delta\Phi}{2D}\right], \tag{10}$$

with

$$k = \frac{1}{\sqrt{DP}} \arctan\left(\frac{D}{\sqrt{aP}}\right), \qquad \Delta\Phi = D\left[\left(1 + \frac{a}{P}\right)\ln\left(\frac{D^2 + aP}{aP}\right) - 1\right]. \tag{11}$$

The occupation probabilities $n_\pm$ of the equivalent bistable system satisfies the following master equation:

$$\begin{bmatrix} \mathrm{d}n_+/\mathrm{d}t \\ \mathrm{d}n_-/\mathrm{d}t \end{bmatrix} = \begin{bmatrix} -N_+(t) & N_-(t) \\ N_+(t) & -N_-(t) \end{bmatrix} \begin{bmatrix} n_+ \\ n_- \end{bmatrix}. \tag{12}$$

Based on the adiabatic elimination theory [3], one can expand equation (9) in series with the small parameter $\mu = [kA\cos(\Omega t)]$, and then combining with equation (12), the expressions of $n_\pm$ can be obtained. The averaged autocorrelation function is given by

$$\begin{aligned} &\langle x(t)x(t+\tau_0)\rangle_{\mathrm{avg}} \\ &\quad = \frac{\Omega}{2\pi}\int_0^{2\pi/\Omega} \lim_{t_0\to-\infty} \big[x_+^2 n_+(t+\tau_0|x_+,t)n_+(t|x_0,t_0) + x_+x_-n_+(t+\tau_0|x_-,t)n_-(t|x_0,t_0) \\ &\qquad + x_+x_-n_-(t+\tau_0|x_+,t)n_+(t|x_0,t_0) + x_-^2 n_-(t+\tau_0|x_-,t)n_-(t|x_0,t_0)\big]\,\mathrm{d}t. \end{aligned} \tag{13}$$

By performing the Fourier transform of the autocorrelation function, one can get the expression of the power spectrum defined for the positive frequency Ω, i.e.,

$$S(\omega) = S_1(\Omega)\delta(\omega - \Omega) + S_2(\omega, \Omega), \tag{14}$$

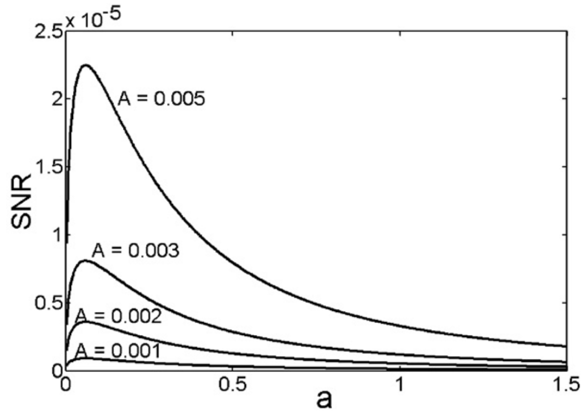

Figure 1. SNR as a function of the system parameter a for $b = 0.3$, $D = 0.75$, $P = 0.2$, $\Omega = 0.01$ with different values of amplitude A.

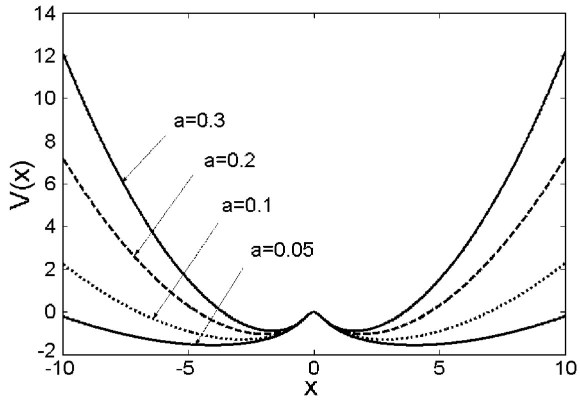

Figure 2. The potential function $V(x)$ for $P = 0.1$, $A = b = 0$, $D = 0.8$ with different values of parameter a.

where

$$S_1(\Omega) = \frac{4\pi D\nu^2}{a(\beta^2 + \Omega^2)}, \qquad S_2(\omega, \Omega) = \frac{4D\beta}{a(\beta^2 + \omega^2)} \left(\frac{1}{\cosh^2(kb)} - \frac{2\nu^2}{\beta^2 + \Omega^2} \right), \tag{15}$$

with

$$\nu = \frac{DAk}{\sqrt{2}\pi \cosh(kb)} \exp\left(-\frac{\Delta\Phi}{2D} \right), \qquad \beta = \frac{\sqrt{2}D \cosh(kb)}{\pi} \exp\left(-\frac{\Delta\Phi}{2D} \right), \tag{16}$$

and k and $\Delta\Phi$ have been defined earlier.

Here $S_1(\Omega)$ is the power density connected with the output signal, $S_2(\omega)$ is the power spectrum associated with the noise background. The signal-to-noise ratio SNR is defined as the ratio between the power density of the signal and the noise background at the frequency $\omega = \Omega$,

$$\mathrm{SNR} = \frac{S_1(\Omega)}{S_2(\omega = \Omega, \Omega)}, \tag{17}$$

where $S_1(\Omega)$ and $S_2(\omega, \Omega)$ have been defined earlier.

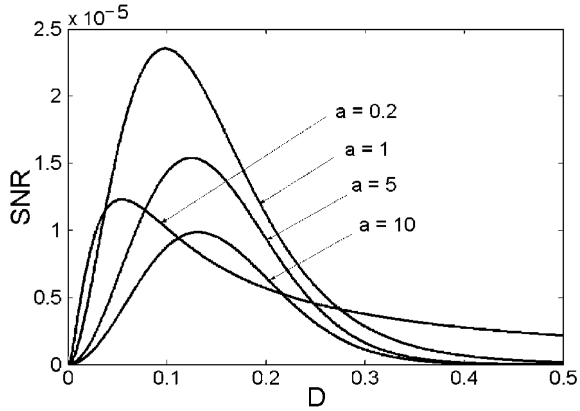

Figure 3. SNR as a function of the multiplicative noise intensity D for $P = 0.1, b = 0.7$, $\Omega = 0.01$, $A = 0.01$ with different values of parameter a.

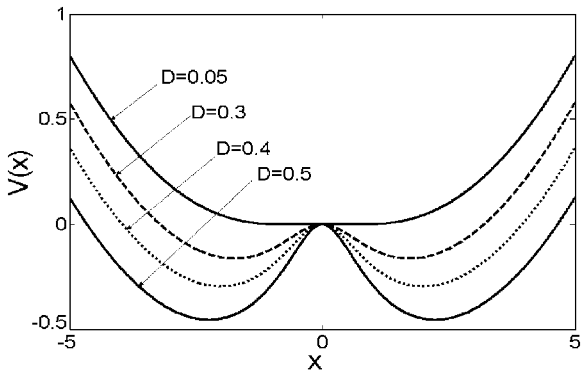

Figure 4. The potential function $V(x)$ for $P = 0.2$, $A = b = 0$, $a = 0.1$ with different values of multiplicative noise intensity D.

3. Discussion and conclusion

Up to now, we have obtained the expression of the signal-to-noise ratio. Now let us discuss the influence of the noise and the system parameters on the signal-to-noise ratio and draw some conclusions.

By virtue of equation (17), the effects of the noise intensity, the system bias b as well as the parameter a on the SNR are discussed through figures 1–8. We analyse these figures by means of the analysis of the system potential.

In figure 1, we analyse the influence of the system parameter a on the SNR. As shown in figure 1, the SNR increases initially as a increases, and then reaches a maximum for some intermediate a. Hence the stochastic resonance in a broad sense takes place. At the same time the SNR increases monotonically with the increase in the amplitude A.

In figure 2, we plot the potential function $V(x)$ for different values of system parameter a and fixed values of the other parameters. As seen in figure 2, the height ΔV of the potential barrier and the distance Δx between the two minima of the potential function decrease with

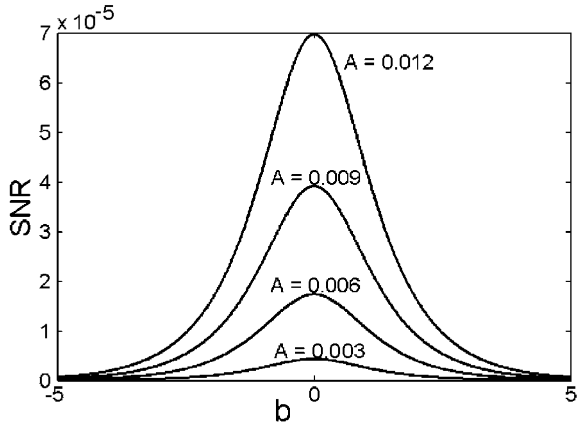

Figure 5. SNR as a function of the system bias b for $a = 2$, $D = 0.8$, $P = 0.5$, $\Omega = 0.01$ with different values of amplitude A.

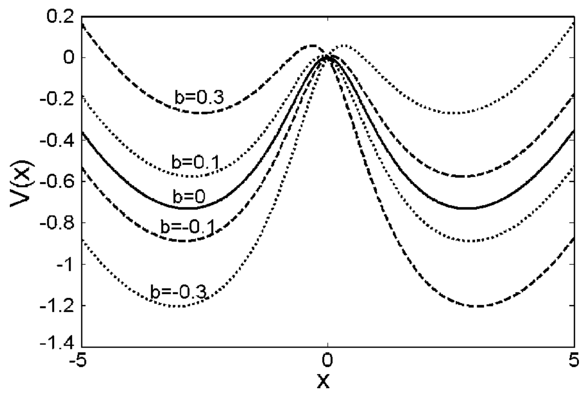

Figure 6. The potential function $V(x)$ for $P = 0.5$, $D = 0.8$, $a = 0.1$, $A = 0$ with different values of bias b.

the increase of the system parameter a. For a very small value of parameter a, the particle moving in the equivalent bistable system can hardly jump over the potential carrier since the height ΔV of the potential is too high. So the particle moves around one of the two potential wells, the output signal is very small, and the SNR is very low. When the parameter a increases, the decrease of ΔV makes the particle easy to jump between the two wells, the output signal can thus be improved, and the SNR increases. On the other hand, for a large value of parameter a, ΔV is relatively low and the distance Δx becomes relatively short, which makes the system turn to be a mono-stable one, the output amplitude thus decreases, and the output noise increases. Therefore, there exists some value of parameter a for which the SNR reaches its maximum value. In other words, the SNR is a non-monotonic function of parameter a.

A multiplicative noise can play a crucial role on the system response. We introduce a multiplicative noise in this paper, and find a non-monotonic behaviour of SNR as a function of the multiplicative noise intensity, which is not mentioned in [21–29]. In figure 3, we show the

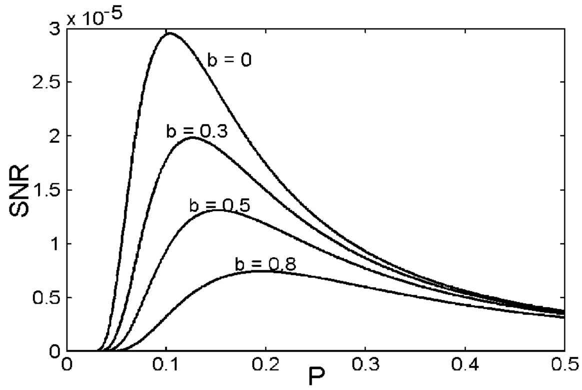

Figure 7. SNR as a function of the additive noise intensity P for $a = 1$, $D = 0.15$, $\Omega = 0.01$, $A = 0.005$ with different values of bias b.

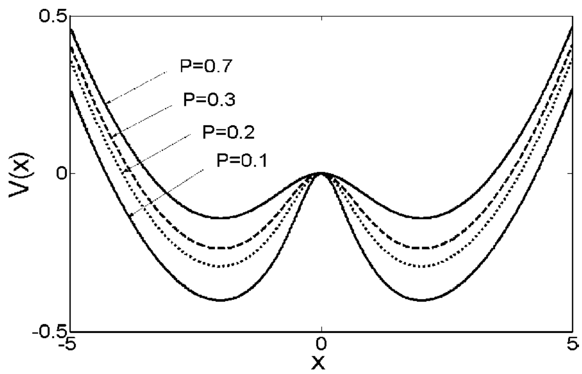

Figure 8. The potential function $V(x)$ for $D = 0.4$, $a = 0.1$, $A = b = 0$ with different values of additive noise intensity P

dependence of the SNR on the multiplicative noise intensity D for different values of system parameter a. As seen in figure 3, the SNR is a non-monotonic function of the multiplicative noise intensity, a maximum exists on the curve of the SNR, i.e., the conventional stochastic resonance occurs. Moreover, the SNR is a non-monotonic function of parameter a, which is consistent with the effect shown in figure 1. As shown in figure 3, the maximum value of SNR for $a = 5$ is higher than those for both $a = 0.2$ and $a = 10$.

As shown in figure 4, the height ΔV of the potential barrier and the distance Δx between the two minima of the potential function decrease with the decrease of the multiplicative noise intensity D, which means that the decrease of the noise strength D has the same effect on the potential height ΔV and the distance Δx as the increase of the parameter a. In fact, for small intensity $D(D < 0.03)$, the particle moves almost in a mono-stable system driven by the additive noise (thermal noise); for large strength $D(D > 0.13)$, the barrier height ΔV turns to be relatively high for the particle to jump over. For both the two cases the output signal becomes very small and the SNR is very low. Therefore, there exists an appropriate multiplicative noise intensity D for which the SNR reaches its maximum value. That is why we see the phenomenon in figure 3.

The system parameter b can also be considered as the asymmetry of the mono-stable potential. We investigate the effect of the asymmetry of the mono-stable system in figure 5. As shown in the figure, the SNR also behaves non-monotonically as the system bias b varies.

We explain figure 5 as follows. As seen in figure 6, the potential barrier ΔV and the distance Δx varies with the increase of the system bias b. One can see that only for $b = 0$, the potential is a symmetric one, while for $b \neq 0$, the potential is an asymmetric one. For fixed values of the other parameters, the barrier height ΔV and the distance Δx are almost equivalent for $b = 0.1$ and $b = -0.1$, for $b = 0.3$ and $b = -0.3$, and so on. Thus the SNR is almost the same for $b = 0.1$ and $b = -0.1$, etc. In addition, one can see that for a positive value of b, a large value of the bias means a high barrier height and a low SNR.

In figure 7, we show the influence of the additive noise intensity P on the SNR for different values of the positive system bias b. We observed clearly the conventional stochastic resonance in the mono-stable system. Moreover, the SNR decreases monotonically with the increasing value of the positive system bias b. The position of the maximum moves to the right with the increment of the parameter b. One can explain the result shown in figure 7 by virtue of figure 8 using the same approach as in figures 1 and 3.

In conclusion, we have studied the stochastic resonance phenomenon in a biased mono-stable system subject to multiplicative and additive noise. The output SNR shows non-monotonic behaviour when it is plotted versus the intensity of the multiplicative and additive noise, as well as the system parameters.

References

[1] Benzi R, Sutera A and Vulpiani A 1981 *J. Phys. A: Math. Gen.* **14** L453
[2] McNamara B *et al* 1988 *Phys. Rev. Lett.* **60** 2626
[3] McNamara B and Weksenfeld K 1989 *Phys. Rev.* A **39** 4854
[4] Dykman M I *et al* 1990 *Phys. Rev. Lett.* **65** 2606
[5] Hu G, Nicolis G and Nicolis C 1990 *Phys. Rev.* A **42** 2030
[6] Zhou T, Moss F and Jung P 1990 *Phys. Rev.* A **42** 3161
[7] Berdichevsky V and Gitterman M 1996 *Europhys. Lett.* **36** 161
[8] Berdichevsky V and Gitterman M 1999 *Phys. Rev.* E **60** 1494
[9] Fauve S and Heslot F 1983 *Phys. Lett.* A **97** 5
[10] Mantegna R N and Spagnolo B 1994 *Phys. Rev. Rap. Commun.E* **49** R1792
[11] Mantegna R N, Spagnolo B and Trapanese M 2001 *Phys. Rev.* E **63** 011101
[12] Stocks N G 2000 *Phys. Rev. Lett.* **84** 2310
[13] Gammaitoni L 1995 *Phys. Rev.* E **52** 4691
[14] Gammaitoni L *et al* 1998 *Rev. Mod. Phys.* **70** 223
[15] Valenti D, Fiasconaro A and Spagnolo B 2004 *Physica* A **331** 477
[16] Jung P and Hanggi P 1982 *Europhys. Lett.* **8** 505
[17] Jung P and Hanggi P 1991 *Phys. Rev.* A **44** 9032
[18] Dykman M I *et al* 1992 *Phys. Rev. Lett.* **68** 2985
[19] Jung P and Hanggi P 1993 *Z. Phys. B: Condens. Matter* **90** 255
[20] Marchesoni F *et al* 1998 *Phys. Lett.* A **248** 332
[21] Dykman M I *et al* 1995 *Nuovo Cimento* D **17** 661
[22] Vilar J M G and Rubi J M 1996 *Phys. Rev. Lett.* **77** 2863
[23] Evstigneev M *et al* 2004 *Europhys. Lett.* **65** 7
[24] Wilson H R and Cowan J D 1972 *Biophys.* **12** 1
[25] Per-Madrid A and Rubi J M 1995 *Phys. Rev.* E **51** 4159
[26] Raikher Y L and Stepanov V I 1995 *Phys. Rev.* B **52** 3493
[27] Lindner J F *et al* 2001 *Phys. Rev.* E **63** 051107
[28] Stocks N G *et al* 1992 *J. Phys. A: Math. Gen.* **25** L1119
[29] Stocks N G *et al* 1993 *J. Phys. A: Math. Gen.* **26** L385